The basic setup[1]

Two players, A and B bargain on how to split a pie of size $\pi > 0$. At time 0, A makes an offer to B. IF B accepts, the game ends and the pie is split according to the offer made. If B rejects, she makes a counteroffer at time $0 + \Delta$. Now A can accept or reject. If A accepts, the game ends. If A rejects, she makes a counteroffer at time $0 + 2\Delta$. Fallback positions for both players are zero.

An offer is a number between 0 and π, hence $\pi-$ the offer is the share of the pie that the responder will get. The key element is that passing of time without reaching an agreement is costly to both players. Suppose that at each moment in time, the payoff by each player is discounted according to the continuous discount factor $e^{-r_i t\Delta}$, where $r_i, i = A, B$ is the discount rate for player i. Denote $\delta_i \equiv e^{-r_i t\Delta}$ for each player $i = A, B$. Each player would like to maximize the share of the pie she gets (her payoff), but the fact that offers can be rejected and passing of time is costly constrains the ability of both players to extract too much from one another.

The most basic question is: does this bargaining problem have a solution? In other words, is there a meaningful equilibrium in the alternating offers game?

Equilibrium

Consider a solution to the game, that is a split of the pie satisfying the following two properties:

1. (*No Delay*) Whenever a player makes an offer, her *equilibrium* offer is accepted by the other player.

2. (*Stationarity*) At an equilibrium, a player makes the same offer whenever she has to make an offer.

To characterize the equilibrium offers, let x_i^* denote the equilibrium offer that player i has whenever she has to make an offer, and suppose that x_i^* satisfies both properties 1 and 2. Suppose that A makes an offer to B, and B rejects. If the equilibrium offer by B is x_B^*, and is accepted by A, then the payoff for B from rejecting *any* offer is $\delta_B x_B^*$. This meets both equilibrium properties.

The fact that players try to maximize their share of the pie has the following implication. For any offer x_A made by A, meaning that B gets $\pi - x_A$, B accepts whenever

$$\pi - x_A > \delta_B x_B^*$$

and rejects whenever

$$\pi - x_A < \delta_B x_B^*.$$

Further, by 1,

$$\pi - x_A \geq \delta_B x_B^*.$$

We want to show that $\pi - x_A = \delta_B x_B^*$. Suppose this is not the case. Then, A can increase her payoff by offering x_A' such that

$$\pi - x_A^* > \pi - x_A' > \delta_B x_B^*$$

meaning that she's gaining at the expenses of B. But this violates 2. Hence, we proved by contradiction that:

$$\pi - x_A^* = \delta_B x_B^* \tag{1}$$

[1]These notes follow Muthoo, 2001. The Economics of Bargaining, in Majumdar, M., editor: Fundamental Economics, in *Encyclopedia of Life Support Systems*, UNESCO.

Symmetrically,

$$\pi - x_B^* = \delta_A x_A^* \tag{2}$$

The system formed by (1) and (2) has a unique solution

$$x_A^* = \pi\left(\tfrac{1-\delta_B}{1-\delta_A\delta_B}\right); \quad x_B^* = \pi\left(\tfrac{1-\delta_A}{1-\delta_A\delta_B}\right) \tag{3}$$

which is the equilibrium of the game. Such equilibrium is obtained by the so-called *backward induction* process, and it is called *subgame–perfect* Nash equilibrium (SPNE). In this case, we can state the following claim:

The unique SPNE of the alternating offers game is a pair of strategies (offers) x_A^*, x_B^* defined as in (3) such that:

- *player A always offers x_A^* and always accepts an offer x_B if and only if $x_B \leq x_B^*$;*
- *player B always offers x_B^* and always accepts an offer x_A if and only if $x_A \leq x^*A$.*

Comments

- If A offers x_A^* at time zero, B will accept, the game ends, and the equilibrium is Pareto-efficient. This is actually the only possible configuration in this game that gives the SPNE and it is rational to play. This is easily proven if players are identical.
- If players are identical, $\delta_A = \delta_B$, and $x^*A = \pi/(1+\delta)$. Further, $\pi - x_A^* = x_B^* = \pi\delta/(1+\delta) < x_A^*$. Hence, A has a *first-mover advantage*. Therefore, rational A won't give this advantage up: she will offer x_A^* at time zero, and nothing less. This way, she prevents B from rejecting and in turn offer $x_B^* = \pi/(1+\delta)$. In this case, A will get $\delta\pi/(1+\delta)$, a smaller share, and will refuse. And so on. Notice that B does not have the first mover advantage.
- If players are not identical, suppose A makes an offer at time zero that is less than x_A^*. B will reject, and will make an offer x_B^*, in which case A gets $\pi - x_B^* = \frac{\pi(1-\delta_A(1+\delta_B))}{1-\delta_A\delta_B}$, always smaller than what she would get by offering x_A^* at time zero. Another way of seeing this is that this sequence of events produces an offer that violates properties 1 and 2 of an equilibrium.
- If A is infinitely patient —that is $\delta_A = 1$, she gets the whole surplus unless B is infinitely patient, too ($\delta_B = 1 - \delta_A$). In this case, the equilibrium is indeterminate. Hence, this model predicts a definite solution only when both players are impatient, and the share each gets depends on their relative time-impatience.
- Consider the following Nash bargaining problem:

  $$\text{Choose } x_A \text{ to maximize } \eta \log x_A + (1-\eta)\log(\pi - x_A)$$

  The solution is $x_A^* = \eta\pi$ which reproduces the Rubinstein's solution if $\eta \equiv (1-\delta_B)/(1-\delta_A\delta_B)$. Hence, the alternating offers model replicates the Nash solution to the bargaining problem, but the players' bargaining power is endogenous in what depends on their discount rates.

Risk of Breakdown

Let us modify the basic setup by assuming that, immediately after any player rejects any offer, with probability $p \in (0,1)$ the negotiations break down in disagreement, whereas with probability

$1-p$ the game proceeds to the next round so that the other player makes the counteroffer. Further, suppose that the *breakdown points* (fallback positions) for A, B are denoted by $\pi > b_A > 0, \pi > b_B > 0$ respectively, and assume further that $b_A + b_B < \pi$. This implies that trade is mutually beneficial, because there is a surplus to be allocated. Suppose now that player i never agrees to an offer, and always rejects. Her payoff is

$$b_i p \sum_{t=0}^{\infty} (1-p)^t$$

which is equal to b_i. This means that player i will not accept anything less than b_i and can guarantee this by always asking at least b_i and rejecting all offers.

It follows that any SPNE of this modified game must involve a payoff greater or equal than b_i for each i. Following the same logic as before, you can show that if player A makes the equilibrium offer x_A^*, player B will always accept an offer

$$\pi - x_A^* \geq p b_B + (1-p) x_B^*$$

and always reject otherwise. The equation tells you that the offer made by A must give B a payoff greater than the maximal expected payoff from the game, which is b_B with probability p and x_B^* with probability $1-p$. As before, we can use 1 and 2 to show that the above condition will hold with equality. Therefore, we have:

$$\begin{aligned} \pi - x_A^* &= p b_B + (1-p) x_B^* && (4) \\ \pi - x_B^* &= p b_A + (1-p) x_B^* && (5) \end{aligned}$$

As before, we can solve this system for x_A^*, x_B^*, to obtain:

$$\begin{aligned} x_A^* &= b_A + \tfrac{1}{2-p}(\pi - b_A - b_B) && (6) \\ x_B^* &= b_B + \tfrac{1}{2-p}(\pi - b_A - b_B) && (7) \end{aligned}$$

If $p = 0$, each player gets her outside option plus $1/2$ of the total surplus generated by the bargain. If $p = 1$, each player i appropriates $\pi - b_j, j \neq i$. This makes sense because if there is 100% probability of breakdown, the only offer by A that B will accept satisfies $\pi - x_A^* = b_B$, which makes B indifferent between the breakdown point and the proposed share of the pie.

A limitation of this approach is that the risk of breakdown is assumed to be exogenous, which in many occasions is not the case.

Exercise How do equations (6) and (7) modify in the presence of discounting? What is the SPNE in this case?